Eurasia Journal of Science and Technology
Print ISSN: 2663-1024
Online ISSN: 2663-1016
DOI: https://doi.org/10.61784/ejst3167

BLOW-UP OF SOLUTIONS FOR FRACTIONAL KIRCHHOFF-TYPE PARABOLIC EQUATIONS WITH LOGARITHMIC TERM

YuXin Chen, YanZi Dong, Ning Pan*, QiZhi Zhang, SiXuan Chen, YuTong Sun, FengFei Wu
College of Science, Northeast Forestry University, Harbin 150040, Heilongjiang, China.
**Corresponding Author: Ning Pan*

Abstract: This paper investigates a class of fractional Kirchhoff-type parabolic equations with a logarithmic term. By means of the potential well theory, we construct the potential energy functional and the Nehari functional, define the stable set and the unstable set, and derive the related lemmas. The blow-up of solutions is discussed under the subcritical energy level and the critical energy level, respectively. Using energy identity, the Sobolev embedding theorem, the intermediate value theorem, the method of contradiction and the convexity method, we prove that the nontrivial weak solution blows up in finite time when the initial energy satisfies the corresponding conditions.
Keywords: Fractional parabolic equation; Kirchhoff type; Potential well; Finite-time blow-up

1 INTRODUCTION

Kirchhoff-type equations originate from the model of elastic string vibration and describe the physical phenomenon that the nonlocal stiffness changes with the overall deformation during the vibration of the string. Such non-local equations break the restriction of classical partial differential equations that depend only on local point information, and can characterize the long-range coupling effects in elastic structures and material deformation more realistically. Therefore, they are widely applied in physics, materials science and other fields [1].

With the continuous improvement of fractional calculus theory, fractional operators, by virtue of their memory effects and non-local features, have been widely introduced into Kirchhoff-type parabolic equations, giving rise to fractional Kirchhoff-type parabolic equations. Compared with integer-order models, fractional models can describe many complex diffusion behaviors that integer-order equations can hardly capture [2-4]. The logarithmic nonlinear term is a kind of slowly growing nonlinear term whose growth rate is weaker than that of polynomial nonlinearities. It changes the geometric structure of the energy functional and brings additional analytical difficulties to the extremum analysis of functionals and the derivation of set invariance under the variational framework [5].

Drawing on the existing research on fractional diffusion equations, this paper considers the following fractional Kirchhoff-type parabolic initial-boundary value problem:

$$\begin{cases} u_t + M\left([u]_s^p\right)(-\Delta)_p^s u + |u|^{p-2}u = |u|^{q-2}u\log|u|, & x\in\Omega, t>0 \\ u(x,t)=0, & x\in\mathbb{R}^n\setminus\Omega, t>0 \\ u(x,0)=u_0(x), & x\in\Omega \end{cases} \tag{1}$$

where $\Omega\subset R^n$ is a bounded smooth domain

$M(m)=m^{\lambda-1}$, $1<\lambda<\frac{p_s^*}{p}$, $p_s^*=\frac{Np}{N-sp}$, $q>p\lambda$ $s\in(0,1)$, $p>2$, $sp<n$.

The following definitions are frequently used in this paper:

Potential energy functional: $J(u)=\frac{1}{p\lambda}\|u\|_{W_0}^{p\lambda}+\frac{1}{p}\|u\|_p^p+\frac{1}{q^2}\|u\|_q^q-\frac{1}{q}\int_\Omega|u|^q\log|u|dx$.

Nehari functional: $I(u)=\|u\|_{W_0}^{p\lambda}+\|u\|_p^p-\int_\Omega|u|^q\log|u|dx$, $I_\delta(u)=\delta\|u\|_{W_0}^{p\lambda}+\|u\|_p^p-\int_\Omega|u|^q\log|u|dx$.

Stable set (bottom region of the well): $W=\{u\in W_0^{s,p}(\Omega)\,|\,J(u(t))<d, I(u(t))>0\}\cup\{0\}$.

Boundary: $\partial W=\{u\,|\,J(u)=d \text{ or } I(u)=0\}$, $W_\delta=\{u\in W_0^{s,p}(\Omega)\,|\,J(u(t))<d, I_\delta(u)>0\}\cup\{0\}$.

Unstable set V (region outside the well): $V=\{u\in W_0^{s,p}(\Omega)\,|\,J(u(t))<d, I(u(t))<0\}$.

Potential well depth: $d=\inf_{u\in N}J(u)$, $V_\delta=\{u\in W_0^{s,p}(\Omega)\,|\,J(u)<d,\ I_\delta(u)<0\}$.

$$N=\{u\in W_0^{s,p}(\Omega)\setminus\{0\}\,|\,I(u)=0\} \tag{2}$$

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$$N_\delta = \{u \in W_0 \mid I_\delta(u) = 0, \| u\|_{W_0} \neq 0\} \tag{3}$$

Definition of continuous embedding of Banach spaces [6]: Let X and Y be Banach spaces; we say that X is embedded into Y ,written as $X \hookrightarrow Y$, if X can be regarded as a subspace of Y , and there exists a positive constant depending only on the spaces X and Y , such that C , $\| x\|_Y \leqslant C\| x\|_X$, $\forall x \in X$. Here C is usually called embedding constant.

Let k be a nonnegative integer, and denote

$C_b^k(\Omega) = \{u \in C^k(\Omega) : \text{The set } D^\alpha u \text{ is bounded above on } \Omega,\ \forall |\alpha| \leqslant k\}$, and equipped with the norm $\| u\|_{C_b^k(\Omega)} = \max\limits_{|\alpha| \leqslant k} \sup\limits_{\Omega} |D^\alpha u|$, then $C_b^k(\Omega)$ is a Banach space, and clearly $C^k(\bar{\Omega}) \hookrightarrow C_b^k(\Omega) \hookrightarrow C^k(\Omega)$.

Theorem (Sobolev embedding theorem) [7]: Assume that $n = 1$.

(1) When Ω is bounded, $W_1^1(\Omega) \hookrightarrow C(\bar{\Omega})$.

(2) When Ω is unbounded and is the union of finitely many open intervals, $W_1^1(\Omega) \hookrightarrow C_b(\Omega) := C_b{}^0(\Omega)$.

(3) When Ω is unbounded and is the union of infinitely many open intervals, $W_1^1(\Omega) \hookrightarrow C(\Omega)$, However, $W_1^1(\Omega) \hookrightarrow C_b(\Omega)$ may not necessarily exist.

Theorem (logarithmic slow growth theorem) [8]: For any given real number $\varepsilon > 0$:

(1) When $|u| \to \infty$, the natural logarithm function grows slower than any positive power function, i.e., $\log|u| = o(|u|^\varepsilon)$.

(2) When $|u| \to 0^+$, the logarithmic singular term grows slower than any negative power function, i.e., $|\log|u|| = o(|u|^{-\varepsilon})$.

2 PRELIMINARY LEMMAS

In this section, we present several lemmas needed for the proofs of the subsequent theorems, investigating the extremum properties of functionals, the characterization of sets, energy estimates, and the invariance of solutions.

Lemma 1. Let $u_0 \in W_0^{s,p}(\Omega)$, then the following conclusions:

(1) $\lim\limits_{t \to 0^+} J(tu) = 0$, $\lim\limits_{t \to +\infty} J(tu) = -\infty$;

(2) There exists a unique $t^* > 0$ satisfying $\dfrac{dJ(tu)}{dt}\Big|_{t=t^*} = 0$;

(3) $J(tu)$ is strictly increasing on $(0,\ t^*)$ and strictly decreasing on $(t^*, +\infty)$;

(4) When $0 < t < t^*$, $I(tu) > 0$; when $t^* < t < +\infty$, $I(tu) < 0$.

Proof. (1) Since $J(tu) = \dfrac{1}{p\lambda} t\| u\|_{W_0}^{p\lambda} + \dfrac{1}{p} t^p \| u\|_p^p + \dfrac{1}{q^2} t^q \| u\|_q^q - \dfrac{1}{q} t^q \| u\|_q^q \log t - \dfrac{1}{q} t^q \int_\Omega |u|^q \log|u|\, dx$, we have $\lim\limits_{t \to 0^+} J(tu) = 0$ and $\lim\limits_{t \to +\infty} J(tu) = -\infty$.

(2)

$$\begin{aligned} \frac{dJ(tu)}{dt} &= t^{p\lambda-1}\| u\|_{W_0}^{p\lambda} + t^{p-1}\| u\|_p^p + \frac{1}{q} t^{q-1} \| u\|_q^q - t^{q-1} \| u\|_q^q \log t - \frac{1}{q} t^{q-1} \| u\|_q^q - t^{q-1} \int_\Omega |u|^q \log|u|\, dx \\ &= t^{p\lambda-1}\| u\|_{W_0}^{p\lambda} + t^{p-1}\| u\|_p^p - t^{q-1} \int_\Omega |u|^q \log|tu|\, dx \\ &= \frac{1}{t} I(tu) \end{aligned} \tag{4}$$

By the definition of $I(u)$, we have $I(tu) = t^q (t^{p\lambda-q}\| u\|_{W_0}^{p\lambda} + t^{p-q}\| u\|_p^p - \int_\Omega |u|^q \log|tu|\, dx)$.

Let $h(t) = t^{p\lambda-q}\| u\|_{W_0}^{p\lambda} + t^{p-q}\| u\|_p^p - \int_\Omega |u|^q \log|tu|\, dx$,

$$h'(t) = (p\lambda - q) t^{p\lambda-q-1}\| u\|_{W_0}^{p\lambda} + (p - q)\ t^{p-q-1}\| u\|_p^p - t^{-1}\| u\|_q^q . \tag{5}$$

Since $q > p\lambda > p > 0$, then $p - p\lambda < 0, p - q < 0$, and thus for all $\forall t > 0$, we have $h'(t) < 0$, and hence $h(t)$ on $(0, +\infty)$ is strictly decreasing $\lim_{t\to 0^+} h(t) = +\infty, \quad \lim_{t\to+\infty} h(t) = -\infty$.

By the intermediate value theorem for continuous functions and the strict monotonicity, there exists a unique $t^* > 0$ such that $J'(t^*u) = 0$; then $\mathrm{I}(t^*u) = 0, \frac{1}{t^*}\mathrm{I}(t^*u) = 0 = \mathrm{J}(\lambda^*u)$, and hence $\exists$ uniquely $t^* > 0$ satisfies $\mathrm{J}(t^*u) = 0$.

(3) From the above properties of $h(t)$, it is strictly decreasing on $(0, +\infty)$, and when $0 < t < t^*$, $h(t) > h(t^*) = 0$, hence $\mathrm{I}(tu) = \mathrm{t}^q h(t) > 0$. By $J'(tu) = \frac{1}{\mathrm{t}^*}\mathrm{I}(tu) > 0$, we know that $\mathrm{J}(tu)$ is strictly increasing on $(0, t^*)$.

When $t > t^*, h(t) < h(t^*) = 0$, so $\mathrm{I}(tu) = \mathrm{t}^q h(t) < 0$. From $J'(tu) = \frac{1}{\mathrm{t}^*}\mathrm{I}(tu) < 0$, we have that $\mathrm{J}(tu)$ is strictly decreasing on $(t^*, +\infty)$.

(4) This follows from (3).

Lemma 2. Let $1 < p < q < p_S^*$,for all $u \in W_0^{s,p}(\Omega) \setminus \{0\}$, $r(\delta) = \left(\frac{\delta e\varepsilon}{S^{q+\varepsilon}}\right)^{\frac{1}{q+\varepsilon-p\lambda}}$,

(1) If $I_\delta(u) < 0$, then $\| u\|_{W_0} > r(\delta)$; in particular, when $I(u) < 0$, we have $\| u\|_{W_0} > r(1)$.

(2) If $I_\delta(u) = 0$, then $\| u\|_{W_0} > r(\delta)$ or $\| u\|_{W_0} = 0$; in particular, when $I(u) = 0$, we have $\| u\|_{W_0} > r(1)$ or $\| u\|_{W_0} = 0$.

(3) If $I_\delta(u) = 0$ and $\| u\|_{W_0} \neq 0$, then when $0 < \delta < q / p\lambda$, we have $J(u) > 0$.

Proof. (1) From $I_\delta(u) < 0$, it is easy to obtain $\| u\|_{W_0} \neq 0$, and hence $\delta\| u\|_{W_0}^{p\lambda} + \| u\|_p^p < \int_\Omega |u|^q \log|u|\, dx$.

Using the inequalities $\log|u| < \frac{|u|^\varepsilon}{e\varepsilon}, t > 0, \varepsilon > 0$, together with the Sobolev embedding theorem, we know that there exists $\exists S > 0$ such that $\| u\|_{q+\varepsilon} \le S\| u\|_{W_0}$, and hence

$$\delta\| u\|_{W_0}^{p\lambda} < \delta\| u\|_{W_0}^{p\lambda} + \| u\|_{\mathrm{p}}^p < \int_\Omega |u|^q \log|u|\, dx < \int_\Omega |u|^{q+\varepsilon} \frac{1}{e\varepsilon} dx = \frac{\| u\|_{q+\varepsilon}^{q+\varepsilon}}{e\varepsilon} < \frac{S^{q+\varepsilon}}{e\varepsilon}\| u\|_{W_0}^{q+\varepsilon} \tag{6}$$

Since $\| u\|_{W_0} \neq 0$, we have $\| u\|_{W_0}^{q+\varepsilon-p\lambda} > \frac{\delta e\varepsilon}{S^{q+\varepsilon}}\delta$, and therefore $\| u\|_{W_0} > \left(\frac{\delta e\varepsilon}{S^{q+\varepsilon}}\right)^{\frac{1}{q+\varepsilon-p\lambda}} = r(\delta)$.

(2) If $\| u\|_{W_0} = 0$, then $u = 0$, substituting into $I_\delta(u)$ gives $I_\delta(u) = 0$; if $\| u\|_{W_0} \neq 0$ and $I_\delta(u) = 0$,

$$\delta\| u\|_{W_0}^{p\lambda} + \| u\|_{\mathrm{p}}^p = \int_\Omega |u|^q \log|u|, \tag{7}$$

similarly, by the same argument as in (1), we obtain

$$\delta\| u\|_{W_0}^{p\lambda} \le \frac{S^{q+\varepsilon}}{e\varepsilon}\| u\|_{W_0}^{q+\varepsilon}, \ \| u\|_{W_0} \ge \left(\frac{\delta e\varepsilon}{S^{q+\varepsilon}}\right)^{\frac{1}{q+\varepsilon-p\lambda}} = r(\delta). \tag{8}$$

(3) If $I_\delta(u) = 0$ and $\| u\|_{W_0} \neq 0$, then substituting $\delta\| u\|_{W_0}^{p\lambda} + \| u\|_{\mathrm{p}}^p = \int_\Omega |u|^q \log|u|\, dx$ into $J(u)$ gives

$$\begin{aligned} J(u) &= \frac{1}{p\lambda}\| u\|_{W_0}^{p\lambda} + \frac{1}{p}\| u\|_{\mathrm{p}}^p + \frac{1}{q^2}\| u\|_{\mathrm{p}}^p - \frac{1}{q}\left(\delta\| u\|_{W_0}^{p\lambda} + \| u\|_{\mathrm{p}}^p\right) \\ &= \left(\frac{1}{p\lambda} - \frac{\delta}{q}\right)\| u\|_{W_0}^{p\lambda} + \left(\frac{1}{p} - \frac{1}{q}\right)\| u\|_{\mathrm{p}}^p + \frac{1}{q^2}\| u\|_{\mathrm{p}}^p. \end{aligned} \tag{9}$$

From $q > p\lambda > p$, we obtain $\frac{1}{p}-\frac{1}{q}>0$; when $\delta \in (0,\frac{q}{p\lambda})$, we have $\left(\frac{1}{p\lambda}-\frac{\delta}{q}\right)\geq 0$. Since $\| u\|_{w_0} \geq r(\delta) > 0$, all terms are strictly greater than 0 , and hence $J(u) > 0$.

Lemma 3. Let $u \in W_0^{s,p}(\Omega)$, and define the potential well depth $d(\delta)=\inf_{u\in N_\delta} J(u)$,

(1) $d(\delta) \geq a(\delta) r(\delta)^{p\lambda}$, where $a(\delta)=\frac{1}{p\lambda}-\frac{\delta}{q}, 0<\delta<q/p\lambda$.

(2) When $0 < \delta \leq 1$, $d(\delta)$ is strictly increasing; when $1\leq\delta\leq\frac{q}{p\lambda}$, $d(\delta)$ is strictly decreasing, and $\delta = 1$ attains its maximum at $d(\delta)$.

Proof. (1) If $u \in N_\delta$, then $I_\delta(u) = 0$, so $\delta\| u\|_{W_0}^{p\lambda}+\| u\|_{\mathrm{p}}^{p} = \int_\Omega |u|^q \log|u|\,dx$,

and hence $J(u)=\left(\frac{1}{p\lambda}-\frac{\delta}{q}\right)\| u\|_{W_0}^{p\lambda}+\left(\frac{1}{p}-\frac{1}{q}\right)\| u\|_{\mathrm{p}}^{p}+\frac{1}{q^2}\| u\|_{\mathrm{p}}^{p}$. Since $q > p\lambda > p$, it follows that $J(u)\geq\left(\frac{1}{p\lambda}-\frac{\delta}{q}\right)\| u\|_{W_0}^{p\lambda} = a(\delta)\| u\|_{W_0}^{p\lambda}$.

By Lemma 2,$\| u\|_{W_0}^{p\lambda} \geq r(\delta)$, hence $J(u) \geq a(\delta) r^{p\lambda}(\delta) > 0$ and $d(\delta)=\inf_{u\in N_\delta} J(u) > 0$.

(2) For $\forall u\in X\setminus\{0\}$, define $\phi(t)=J(tu), t>0, I_\delta(tu)=\delta t^{p\lambda}\| u\|_{W^{1,p}}^{p}+t^{p}\| u\|_{p}^{p}-t^{q}\int_\Omega|u|^q\log|u|dx$.

If $tu\in N_\delta$, then $I_\delta(tu)=0$. For each $\delta > 0$, there exists a unique $t(\delta;u)>0$ such that $t(\delta;u)u \in N_\delta$.

Hence $d(\delta)=\inf_{u\in N_\delta} J(t(\delta;u)u)$. From the condition $I_\delta(tu)=0$ we obtain

$$\delta\| u\|_{W^{1,p}}^{p}t^{p\lambda}+\| u\|_{p}^{p}t^{p} = t^{q}\int_\Omega|u|^q\log|u|dx \tag{10}$$

$q > p\lambda > p, q-p\lambda>0, q-p>0,$. Differentiating the implicit function in (10) with respect to δ , we obtain

$$\|u\|_{W_0}^{p\lambda}t^{p\lambda}+\delta\|u\|_{W_0}^{p\lambda}p\lambda t^{p\lambda-1}\frac{dt}{d\delta}+p\|u\|_{p}^{p}t^{p-1}\frac{dt}{d\delta}=t^{q-1}\|u\|_{2}^{2}\frac{dt}{d\delta}+qt^{q-1}\frac{dt}{d\delta}\int_\Omega|u|^q\log|ut|dx \tag{11}$$

$$\frac{dt}{d\delta}(\delta\| u\|_{W_0}^{p\lambda}p\lambda t^{p\lambda-1}+pt^{p-1}\| u\|_{p}^{p}-qt^{q-1}\int_\Omega|u|^q\log(tu)dx-t^{q-1}\| u\|_{q}^{q}) = -t^{p\lambda}\| u\|_{W_0}^{p\lambda} \tag{12}$$

Combining (10), we know that $\int_\Omega|u|^q\log|tu|dx=\frac{\delta\| u\|_{W_0}^{p\lambda}t^{p\lambda}+\| u\|_{p}^{p}t^{p}}{t^q}$, and substituting (10) into the above expression gives $\frac{dt}{d\delta}=\frac{(p\lambda-q)\delta\| u\|_{W_0}^{p\lambda}t^{p\lambda-1}+(p-q)\| u\|_{p}^{p}t^{p-1}}{-t^{p\lambda}\| u\|_{W_0}^{p\lambda}}>0$.

We know that $J(tu)=\frac{1}{p\lambda}t^{p\lambda}\| u\|_{W_0}^{p\lambda}+\frac{1}{p}t^{p}\| u\|_{p}^{p}+\frac{1}{q^2}t^{q}\| u\|_{q}^{q}-\frac{1}{q}t^{q}\int_\Omega|u|^q\log(tu)dx,$, and

Substituting (10) into $J(tu)$ yields: $J(tu)=\left(\frac{1}{p\lambda}-\frac{\delta}{q}\right)t^{p\lambda}\| u\|_{W^{1,p}}^{p}+\left(\frac{1}{p}-\frac{1}{q}\right)t^{p}\| u\|_{p}^{p}+\frac{1}{q^2}t^{q}\| u\|_{q}^{q}.$

Now we analyze the function $d(\delta)=\inf_{u\neq0} J(t(\delta;u)u)$:

When $\delta\in(0,1)$, as δ increases, $\frac{1}{p\lambda}-\frac{\delta}{q}$ decreases, but $t(\delta;u)$ increases strictly, and the $t(\delta)$ growth effect dominates, so $J(t(\delta;u)u)$ is strictly increasing. Taking the infimum over $J(tu)$, $d(\delta)=\inf_{u\in N_\delta} J(t(\delta;u)u)$ is also strictly increasing.

When $\delta\in[1,\frac{q}{p\lambda})$, as δ increases, $\frac{1}{p\lambda}-\frac{\delta}{q}$ decreases, while $t(\delta;u)$ still increases, but the decay effect of $\frac{1}{p\lambda}-\frac{\delta}{q}$ overpowers the growth effect, $\delta\to\left(\frac{q}{p\lambda}\right)^{-}$, $\left(\frac{1}{p\lambda}-\frac{\delta}{q}\right)\to 0$. Therefore, $J(t(\delta;u)u)$ decreases strictly, and taking the infimum, $d(\delta)=\inf_{u\in N_\delta}J(t(\delta;u)u)$ is also strictly decreasing.

$d(\delta)$ is strictly increasing on $(0,1)$ and strictly decreasing on $\left(1,\frac{q}{p\lambda}\right)$, so $\delta=1$ is the unique maximizer, and hence $\delta=1$ attains its maximum.

Lemma 4. Let $u\in W_0^{s,p}(\Omega)$ satisfy $0<J(u)<d$, and let δ_1,δ_2 be the two roots of $d(\delta)=J(u)$. Then the sign of $I_\delta(u)$ is unchanged on the interval (δ_1,δ_2) .

Proof. By contradiction, suppose that $I_\delta(u)$ changes sign on $\delta\in(\delta_1,\delta_2)$. Then one can take $\bar{\delta}\in(\delta_1,\delta_2)$ such that $I_{\bar{\delta}}(u)=0$.

Since $I_{\bar{\delta}}(u)=0$ and $u\neq 0$, we have $u\in N_{\bar{\delta}}$. $\because d(\delta)=\inf_{w\in N_\delta}J(w)$, hence $J(u)>d(\bar{\delta})$. This contradicts $J(u)=d(\delta_1)=d(\delta_2)<d(\bar{\delta})$. Therefore, the assumption is false, and the sign remains unchanged.

Definition 5. Let $u(x,t)$ be a weak solution of (1.1) ,

(1) If $0\le t<\infty$ exists on $u(x,t)$, then $T_{\max}=\infty$.

(2) If there exists a $t_0\in(0,\infty)$ such that $u(x,t)$ exists on $0\le t<t_0$, but does not exist when $t=t_0$; then $T_{\max}=t_0$.

The $T_{\max}$ satisfying the above condition is called the maximal existence time of the solution.

Lemma 6. If $u(t)$ is a local weak solution, then for all $\forall t\in[0,T_{\max})$, $J(u(t))+\int_0^t\|u_\tau\|_2^2d\tau=J(u_0)$.

Proof. Multiplying equation (1) by u_t and integrating over Ω , we obtain

$$\int_\Omega u_t^2dx+\|u\|_{W_0}^{p(\lambda-1)}\langle(-\Delta)_p^s u,u_t\rangle+\int_\Omega|u|^{p-2}u\cdot u_tdx-\int_\Omega|u|^{q-2}u\log|u|\cdot u_tdx=0. \tag{13}$$

Furthermore, $\frac{d}{dt}J(u)=\|u\|^{p(\lambda-1)}\langle(-\Delta)_p^s u,u_t\rangle+\int_\Omega|u|^{p-2}u\cdot u_tdx-\int_\Omega|u|^{q-2}u\log|u|\cdot u_tdx$,

where $\left\langle(-\Delta)_p^s u,u_t\right\rangle=\iint_{R^{2n}}\frac{|u(x)-u(y)|^{p-2}\left(u(x)-u(y)\right)\left(u_t(x)-u_t(y)\right)}{|x-y|^{n+sp}}dxdy$. Substituting $\frac{d}{dt}J(u)$ into the above equation yields $\frac{d}{dt}J(u(t))+\int_\Omega u_t{}^2dx=\frac{d}{dt}J(u)+\|u_t\|_2^2=0$.

Integrating t over $(0,t_0)$ once more, $J(u(t))-J(u_0)+\int_0^{t_0}\|u_\tau(\tau)\|_2^2d\tau=0$ we obtain $J(u(t))+\int_0^{t_0}\|u_\tau\|_2^2d\tau=J(u_0)$.

Lemma 7. Let $u_0\in W_0^{s,p}(\Omega)$ and satisfy the initial energy $0<J(u_0)\le e<d$; let $\delta_1<\delta_2$ be $d(\delta)=e$ the two roots of the equation, and if at the initial time $I(u_0)<0$, then for $\forall\delta\in(\delta_1,\delta_2)$ problem (1.1) of the local weak solution $u(t)$, on its maximal existence time $T_{\max}$, such that whenever $t\in(0,T_{\max})$, we always have $u(t)\in V_\delta$.

Proof. Since $I(u_0)<0$ and on (δ_1,δ_2) we have $\delta=1$, by Lemma 4, for all $\forall\delta\in(\delta_1,\delta_2)$, $I_\delta(u_0)<0$. Moreover, since $J(u_0)\le e<d(\delta)$, it follows that $u_0\in V_\delta$.

We next prove that $u(t)\in V_\delta$. By contradiction, in view of the time continuity of $I(u)$, suppose that $\exists t_0\in(0,T)$, $\delta_0\in(\delta_1,\delta_2)$, so that when $0\le t<t_0$, $u(t)\in V_\delta$, , but when $t=t_0$, $u(t_0)\in\partial V_{\delta_0}$. That is, $I_{\delta_0}(u(t_0))=0$ or $J(u(t_0))=d(\delta_0)$. By Lemma 6, substituting $t=t_0$ into the energy identity, we obtain $J(u(t_0))=J(u_0)-\int_0^{t_0}\|u_\tau\|_2^2d\tau\le J(u_0)$.

Since initially $J(u_0) \le e$, it follows that $J(u(t_0)) \le J(u_0) < \mathrm{e}$, and $\delta_0 \in (\delta_1, \delta_2)$, $d(\delta_0) > e$, so $J(u(t_0)) < d(\delta_0)$. That is, $J\left(u\left(t_0\right)\right) \neq d\left(\delta_0\right)$, which excludes the possibility of $J(u(t_0)) = d(\delta_0)$.

Therefore, if $I_{\delta_0}\left(u\left(t_0\right)\right) = 0$, then by Lemma 2, if $I_\delta\left(u\left(t_0\right)\right) = 0$ we know $u(t_0) \in N_\delta$, and hence $u\left(t_0\right) \neq 0$, and $\| u \|_{W_0}^{p\lambda} \ge r_\delta > 0$. Moreover, by $I_{\delta_0}\left(u\left(t_0\right)\right) = 0, u\left(t_0\right) \in N_\delta$, this contradicts $d\left(\delta\right) = \inf_{u \in N_\delta} J\left(u\right), J\left(u\left(t_0\right)\right) \ge d(\delta)$. $J(u(t_0)) < d(\delta)$.

Therefore, the assumption is false; that is, on $\forall t \in \left[0, T_{\max}\right)$, we always have $u\left(t\right) \in V_\delta$.

Lemma 8. Let $u_0 \in W$, i.e., $J\left(u_0\right) < d$, $I\left(u_0\right) > 0$ or $u_0 = 0$. If $u\left(t\right)$ is a local weak solution of problem (1), then for any $t \in [0, T_{\max})$, we always have $u\left(t\right) \in W$.

Proof. If $u_0 = 0$, then clearly $u\left(t\right) = 0$. The conclusion holds.

Now suppose that $u_0 \neq 0$, and $J\left(u_0\right) < d, I(u_0) > 0$. By Lemma 7, for all $\forall t \in \left[0, T_{max}\right)$, we have $J\left(u\left(t\right)\right) + \int_0^t \| u_\tau(\tau) \|^2 d\tau = J\left(u_0\right)$, hence $J\left(u\left(t\right)\right) \le J\left(u_0\right) < d$, and $\forall t \in \left[0, T_{max}\right)$.

We now prove that for all $t \in \left[0, T_{max}\right)$, $I\left(u\left(t\right)\right) > 0$ holds. Otherwise, there exists $\exists t_0 \in \left(0, T_{max}\right)$ such that $I\left(u\left(t_0\right)\right) \le 0$. Since $I\left(u\left(t\right)\right)$ is continuous with respect to t, and $I\left(u_0\right) > 0$, there exists a smallest $t_1 \in \left(0, t_0\right]$ such that $I\left(u\left(t_1\right)\right) = 0$.

If $u\left(t_1\right) = 0$, then clearly $u\left(t_1\right) \in W$, and it cannot escape from the stable set. Therefore, it suffices to consider $u\left(t_1\right) \neq 0$. In this case $u\left(t_1\right) \in N = \{u \in W_0^{s,p}\left(\Omega\right) \setminus \{0\} \mid I\left(u\right) = 0\}$. From $d = \inf_{u \in N} J\left(u\right)$, we get $J\left(u\left(t_1\right)\right) \ge d$, which contradicts the previous $J\left(u\left(t_1\right)\right) < d$. Hence, for $\forall t \in \left[0, T_{max}\right)$, we must have $I\left(u\left(t\right)\right) > 0$. Combining with $J\left(u\left(t\right)\right) < d$, we obtain $u\left(t\right) \in W$.

3 FINITE-TIME BLOW-UP UNDER THE SUBCRITICAL ENERGY LEVEL

Theorem 9 (Finite-time blow-up under the subcritical energy level). Let $u_0 \in W_0$, $J(u_0) < 0$ and $I(u_0) < 0$. Then the solution $u(t)$ of the equation blows up in finite time; that is, there exists a time $T > 0$ such that $\lim_{t \to T} \int_0^t \| u(\tau) \|_2^2 \mathrm{d}\tau = +\infty$.

Proof. Define the time-integral function of the L^2 norm of the solution as $M(t) = \int_0^t \| u(\tau) \|_2^2 \mathrm{d}\tau$.

We argue by contradiction and suppose that $T = \infty$. Taking the first derivative of t, we have $M'(t) = \| u(t) \|_2^2 \ge 0$; taking the second t second derivative, $M''(t) = 2(u, u_t) = 2\int_\Omega u u_t \mathrm{d}x = -2\langle J'(u), u \rangle = -2\left(\| u \|_{W_0}^{p\lambda} + \| u \|_p^p - \int_\Omega |u|^q \log |u| \, \mathrm{d}x \right) = -2I(u)$.

Moreover, since $J(u) - \frac{1}{q} I(u) = \left(\frac{1}{p\lambda} - \frac{1}{q} \right) \| u \|_{W_0}^{p\lambda} + \left(\frac{1}{p} - \frac{1}{q} \right) \| u \|_p^p + \frac{1}{q^2} \| u \|_q^q$, we have

$$-I(u) = -qJ(u) + \frac{q - p\lambda}{p\lambda} \| u \|_{W_0}^{p\lambda} + \frac{q - p}{p} \| u \|_p^p + \frac{1}{q} \| u \|_q^q . \tag{14}$$

Since $q > p\lambda > p > 2$, which is bounded by $-I(u) \ge -qJ(u) + \frac{q - p\lambda}{p\lambda} \| u \|_{W_0}^{p\lambda}$; substituting it into $M''(t)$, we have $M''(t) \ge -2qJ(u) + \frac{2(q - p\lambda)}{p\lambda} \| u \|_{W_0}^{p\lambda}$. Combining with the energy identity $J(u(t)) = J(u_0) - \int_0^t \| u_\tau \|_2^2 \mathrm{d}\tau$, we obtain $M''(t) \ge \frac{2(q - p\lambda)}{p\lambda} \| u \|_{W_0}^{p\lambda} + 2q \int_0^t \| u_\tau \|_2^2 \mathrm{d}\tau - 2qJ(u_0)$.

Using the Sobolev embedding and the basic inequality $S \le S^{\alpha}+1, S \ge 0, \alpha \ge 1$: there exists a constant $\exists$ such that $C>0 \ \| u\|_{W_0} \ge C\| u\|_2$ and

$$M''(t) \ge 2q\int_0^t \| u_\tau\|_2^2 \mathrm{d}\tau + \frac{2(q-p\lambda)}{p\lambda}\| u\|_{W_0}^2 - \left(2qJ(u_0)+\frac{2(q-p\lambda)}{p\lambda}\right) \tag{15}$$

$$M''(t) \ge 2q\int_0^t \| u_\tau\|_2^2 \mathrm{d}\tau + \frac{2C(q-p\lambda)}{p\lambda}\| u\|_2^2 - \left(2qJ(u_0)+\frac{2(q-p\lambda)}{p\lambda}\right). \tag{16}$$

From the above, we know that $M'(t)=\| u\|_2^2$; substituting it into $M''(t)$, we have

$$M''(t) \ge 2q\int_0^t \| u_\tau\|_2^2 \mathrm{d}\tau + \frac{2C(q-p\lambda)}{p\lambda}M'(t) - \left(2qJ(u_0)+\frac{2C(q-p\lambda)}{p\lambda}\right) \tag{17}$$

By the Cauchy-Schwarz inequality, $\left(\int_0^t\int_\Omega uu_\tau \mathrm{d}x\mathrm{d}\tau\right)^2 \le \left(\int_0^t \| u\|_2^2\mathrm{d}\tau\right)\left(\int_0^t \| u_\tau\|_2^2\mathrm{d}\tau\right) = M(t)\int_0^t \| u_\tau\|_2^2\mathrm{d}\tau$, substituting $\int_0^t\int_\Omega u_\tau u dx d\tau = \frac{1}{2}M^{'}(t)-\frac{1}{2}M^{'}(0)$ into the above equation and multiplying both sides by $2q$, we obtain

$$2qM(t)\int_0^t \| u_\tau\|_2^2\mathrm{d}\tau \ge \frac{q}{2}\left(M'(t)\right)^2 - qM'(0)M'(t) \tag{18}$$

Multiplying both sides of (17) by $M(t)$:

$$M''(t)M(t) \ge 2qM(t)\int_0^t \| u_\tau\|_2^2\mathrm{d}\tau + \frac{2C(q-p\lambda)}{p\lambda}M'(t)M(t) - \left(2qJ(u_0)+\frac{2(q-p\lambda)}{p\lambda}\right)M(t) \tag{19}$$

Substituting (18) into the above equation, we obtain

$$\begin{aligned} M''(t)M(t)-\frac{q}{2}\left(M'(t)\right)^2 &\ge \frac{2C(q-p\lambda)}{p\lambda}M(t)M'(t) - q\| u_0\|_2^2 M'(t) - \left(2qJ(u_0)+\frac{2(q-p\lambda)}{p\lambda}\right)M(t) \\ &\ge M(t)M'(t)\left(\frac{2C(q-p\lambda)}{p\lambda} - \frac{q\| u_0\|_2^2}{M(t)} - \frac{2qJ(u_0)+\frac{2(q-p\lambda)}{p\lambda}}{M'(t)}\right) \end{aligned} \tag{20}$$

Then for $t\to 0$, the second and third terms on the right-hand side both tend to zero, whereas the constant term $\frac{2C(q-p\lambda)}{p\lambda}>0$.

Case (I): If $J(u_0)<0$, then $-2qJ(u_0)\ge 0$,

$$M(t)M''(t)-\frac{q}{2}\left(M'(t)\right)^2 \ge \frac{2C(q-p\lambda)}{p\lambda}M(t)M'(t) - q\| u_0\|_2^2 M'(t) - \frac{2(q-p\lambda)}{p\lambda}M(t) \tag{21}$$

We now prove that on $t>0$, $I(u)<0$. If this conclusion were false, there would exist some $t_0>0$ such that $I(u(t_0))=0$. Then on $0<t<t_0$, we have $I(u)<0$. By Lemma 2 (1), we obtain on $0\le t<t_0$ that $\| u\|_{W_0}>r(1)$, as well as $J(u(t))\ge d$, which contradicts the energy decay; hence, when $t>0, I(u)<0$.

From $M''(t)=-2I(u)$, we get on $t\ge 0$ that $M''(t)>0$. Therefore, $M(t)$ is strictly convex and $M'(t)$ is monotonically increasing. For $t\ge t_0>0, M'(t)>M'(t_0)$ and $M(t_0)=\int_0^{t_0}\| u\|_2^2 d\tau>0$, we have:

$$M(t)\ge M'(t_0)(t-t_0)+M(t_0)>M'(0)(t-t_0). \tag{22}$$

Moreover, since $M'(0)=\| u\|_2^2$, in view of formula (21), for sufficiently large t , we must have $\frac{C(q-p\lambda)}{p\lambda}M(t)>q\| u_0\|_2^2$ and $\frac{C(q-p\lambda)}{p\lambda}M'(t)>\frac{2(q-p\lambda)}{p\lambda}$; substituting into inequality (21) yields $M(t)M''(t)-\frac{q}{2}\left(M'(t)\right)^2>0$.

Case (II): If $0\le J(u_0)<d$ and $I(u_0)<0$.

Since $J(u_0)<d$ and $I(u_0)<0$, we have $\forall t>0$ for all $J(u(t))\le J(u_0)<d$. We next prove that $I(u(t))<0$. If the assumption fails, then there exists $\exists t^*$ such that $I(u(t^*))=0$. By the definition of the potential well depth d $d=\inf_{u\in W}J(u)$, we have $J(u(t^*))\ge\inf_{u\in W}J(u)=d$, which contradicts $J(u(t^*))<d$; hence, for $\forall t>0$, we always have $I(u(t))<0$.

Furthermore, there exists a positive constant $\eta>0$ such that $-I(u(t))\ge\eta>0$, and hence $M''(t)=-2I(u(t))>0$,

$$M'(t)=\int_0^t M''(\tau)\mathrm{d}\tau+M'(0)\ge 2\eta t+M'(0) \tag{23}$$

$$M(t)=\int_0^t M'(\tau)\mathrm{d}\tau+M(0)\ge \eta t^2+\| u_0\|_2^2 t+M(0) \tag{24}$$

When $t\to+\infty$, we must have $M'(t),M(t)\to+\infty$. By

$$M''(t)M(t)-\frac{q}{2}\left(M'(t)\right)^2\ge M(t)M'(t)\left(\frac{2C(q-p\lambda)}{p\lambda}-\frac{q\| u_0\|_2^2}{M(t)}-\frac{2qJ(u_0)+\frac{2(q-p\lambda)}{p\lambda}}{M'(t)}\right) \tag{25}$$

By $\frac{2C(q-p\lambda)}{p\lambda}>0$ and $M(t)\to+\infty$, $M'(t)\to+\infty$, there exists a sufficiently large $t_0>0$ such that

$$M''(t)M(t)-\frac{q}{2}\left(M'(t)\right)^2>0. \tag{26}$$

Define $y(t)=M(t)^{1-\frac{q}{2}}$, and by $M(t)M''(t)-\frac{q}{2}\left(M'(t)\right)^2>0, y''(t)=\left(1-\frac{q}{2}\right)M(t)^{-\frac{q}{2}}\left(M(t)M''(t)-\frac{q}{2}\left(M'(t)\right)^2\right)<0.$

That is, $y'(t)$ is monotonically decreasing and negative, so $y'(t)\le y'(t_0)=-c<0$ holds for all $t\ge t_0$. Integrating the above relation yields $y'(t)\le y'(t_0)=-c(t-t_0)$, whose right-hand side must drop to zero at some finite time $T=t_0+\frac{y(t_0)}{c}$; since $y(t)\ge 0$, it can only happen that when $t\to T^-$, $M(t)\to+\infty$, i.e., $\int_0^T\| u_0\|_2^2 dt=+\infty$ diverges.

4 FINITE-TIME BLOW-UP UNDER THE CRITICAL ENERGY LEVEL

Theorem 10 (Finite-time blow-up under the critical energy level). Let $J(u_0)=d$ $u_0\in W_0^s(\Omega)$, initial energy $J(u_0)=d$ and $I(u_0)<0$. Then any nontrivial weak solution of the corresponding equation necessarily blows up in finite time.

Proof. We argue by contradiction and assume that the solution exists globally, i.e., the maximal existence time $T=\infty$.

Define the auxiliary function $M(t)=\int_0^t\| u(\tau)\|_2^2\mathrm{d}\tau$; by Theorem 9 and $J(u_0)=d$, we obtain

$$M(t)M''(t)-\frac{q}{2}\left(M'(t)\right)^2\ge\frac{2C(q-p\lambda)}{p\lambda}M(t)M'(t)-q\| u_0\|_2^2M'(t)-M(t)\left(2qd+\frac{2(q-p\lambda)}{p\lambda}\right) \tag{27}$$

Since at the initial time $J(u_0)=d>0$ and $I(u_0)<0$, by the continuity of the solution map $t\mapsto J(u(t)), t\mapsto I(u(t))$, there must exist a sufficiently small time $t_1>0$ such that on $0\le t\le t_1$, we have $J(u(t))>0$ and $I(u(t))<0$. Because on $[0,t_1]$, $I(u(t))<0$, we have $\int_\Omega uu_t\mathrm{d}x=-I(u)>0$, and hence for $u_t\not\equiv 0$, on $[0,t_1]$, we have $\int_0^{t_1}\| u_\tau\|_2^2\mathrm{d}\tau>0$.

Using the energy identity, we compute energy at time t_1 as

$$J(u(t_1))=J(u_0)-\int_0^{t_1}\| u_\tau\|_2^2\mathrm{d}\tau=d-\int_0^{t_1}\| u_\tau\|_2^2\mathrm{d}\tau<d\,, \tag{28}$$

Treating $t=t_1$ as the initial value we then have $J(u(t_1))<d$, which reduces to the subcritical case. Hence, we obtain $\exists\delta\in(\delta_1,\delta_2)$, $t_1\le t<\infty$, and $u(t)\in V_\delta$, where (δ_1,δ_2) is the maximal interval on which $\delta=1$ holds when $d(\delta)>d_1$. Therefore, on $\delta\in(1,\delta_2)$, $t_1\le t<\infty$, we have $I_\delta(u)<0$ and $\| u\|_{W_0}>r(\delta)$; moreover, on $t_1\le t<\infty$, we have $I_{\delta_2}(u)\le 0$, $\| u\|_{W_0}>r(\delta_2)$, and hence

$$M''(t)=-2I(u(t))\ge 2(\delta_2-1)\| u\|_{W_0}^{p\lambda}\ge 2(\delta_2-1)r^{p\lambda}(\delta_2)=C(\delta_2)>0,\forall t\ge t_1 \tag{29}$$

and

$$M'(t)\ge C(\delta_2)(t-t_1)+M'(t_1),\quad t\ge t_1 \tag{30}$$

and also

$$M(t)\ge\frac{1}{2}C(\delta_2)(t-t_1)^2+M'(t_1)(t-t_1)+M(t_1),\quad t\ge t_1 \tag{31}$$

Therefore, when $t\to+\infty$, we have $u'(t)\to\infty$ and $u(t)\to\infty$.

From the inequality

$$M(t)M''(t)-\frac{q}{2}\left(M'(t)\right)^2\ge M(t)M'(t)\left[\frac{2C(q-p\lambda)}{p\lambda}-\frac{q\| u_0\|_2^2}{M(t)}-\frac{2qd+\dfrac{2(q-p\lambda)}{p\lambda}}{M'(t)}\right] \tag{32}$$

When t is sufficiently large, the denominator $M(t)$ and $M'(t)\to+\infty$, so the last two terms tend to zero. Moreover, since $\dfrac{2C(q-p\lambda)}{p\lambda}>0$ is a positive constant, there exists a sufficiently large time $T_0\ge t_1$ such that for all $t\ge T_0$, the right-hand side of the inequality is strictly greater than 0 : $M(t)M''(t)-\frac{q}{2}\left(M'(t)\right)^2>0.$

Define the auxiliary function

$$y(t)=\left(M(t)\right)^{1-\frac{q}{2}}. \tag{33}$$

$$y''(t)=\left(1-\frac{q}{2}\right)\left(M(t)\right)^{-\frac{q}{2}-1}\left[M(t)M''(t)-\frac{q}{2}\left(M'(t)\right)^2\right] \tag{34}$$

Since $q>p\lambda>2$, it follows that $1-\dfrac{q}{2}<0$, and hence for $t\ge T_0$, $y''(t)\le 0$ holds.

Since $y(t)>0$ and it is strictly convex, there exists $T^*<\infty$ such that $\lim_{t\to T^*}y(t)=0$, i.e., $\lim_{t\to T^*}M(t)=+\infty$. This contradicts the assumption, and the proof is complete.

5 CONCLUSION

In this paper, for a class of fractional Kirchhoff-type parabolic equations with a logarithmic nonlinear term, we employ the potential well method to construct the potential energy functional and the Nehari functional, and establish the related lemmas that characterize the stable set, the unstable set and the invariance of solution orbits. Finally, we prove that, under the subcritical and the critical initial energy levels respectively, when the initial data fall into the unstable set, the weak solutions blow up in finite time.

COMPETING INTERESTS

The authors have no relevant financial or non-financial interests to disclose.

FUNDING

The project was supported by Undergraduate Innovation Training Program of Northeast Forestry University (No. 202610225493).

ACKNOWLEDGEMENTS

We sincerely thank Teacher Ning Pan for her careful guidance on this paper. From establishing the research idea and the theoretical derivation to revising and improving the paper, she gave us meticulous advice and valuable suggestions, which enabled us to complete this research. We also thank our team members YuXin Chen, YanZi Dong, SiXuan Chen, YuTong Sun and FengFei Wu. During the research, we divided the work and cooperated with each other, discussed problems and verified the proofs together, and jointly overcame the difficulties encountered in the research to complete this paper.

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